

Quantum computation with Kerr-nonlinear photonic crystals

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Abstract

In this paper, we consider a method for implementing a quantum logic gate with photons whose wave function propagates in a one-dimensional Kerr-nonlinear photonic crystal. The photonic crystal causes the incident photons to undergo Bragg reflection by its periodic structure of dielectric materials and forms the photonic band structure, namely, the light dispersion relation. This dispersion relation reduces the group velocity of the wave function of the photons, so that it enhances nonlinear interaction of the photons. (Because variation of the group velocity against the wave vector is very steep, we have to tune up the wavelength of injected photons precisely, however.) If the photonic crystal includes layers of a Kerr medium, we can rotate the phase of the wave function of the incident photons by a large angle efficiently. We show that we can construct the nonlinear sign-shift (NS) gate proposed by Knill, Laflamme, and Milburn (KLM) by this method. Thus, we can construct the conditional sign-flip gate for two qubits, which is crucial for quantum computation. Our NS gate works with probability unity in principle while KLM's original one is a nondeterministic gate conditioned on the detection of an auxiliary photon.

1 Introduction

Since Shor's quantum algorithms for prime factorization and discrete logarithms appeared, many researchers have been devoting their efforts to realization of quantum computation in their laboratory [1]. To implement a quantum computer, we have to prepare qubits, which are two-state systems, and quantum logic gates, which apply unitary transformations to qubits. Generally speaking, we can apply an arbitrary transformation of $U(2)$

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group to a qubit at ease no matter which physical system we choose as the qubit. However, almost all the researchers think implementation of a two-qubit gate to be very difficult because the two-qubit gate has to generate entanglement between two local systems. Moreover, it is shown that we can construct any unitary transformation applied to qubits from $U(2)$ transformations and a certain two-qubit gate, such as the controlled-NOT gate or the conditional sign-flip gate [2, 3, 4]. From the above reasons, theoretical and experimental physicists aim for realizing the two-qubit gate.

Among proposals made for implementing quantum computation, for example realization of qubits by cold trapped ions interacting with laser beams [5, 6], polarized photons in the cavity quantum electrodynamics system [7], nuclear spins of molecules under nuclear magnetic resonance [8], and so on, the scheme of Knill, Laflamme, and Milburn (KLM) is very unique [9, 10]. They show a method for constructing the conditional sign-flip gate from linear optical elements (single photon sources, beamsplitters, and photodetectors). Because many researchers believe that only nonlinear interaction between two qubits can generate quantum correlation, KLM's scheme that does not require any nonlinear devices seems to be novel and attractive. In KLM's method, Bose-Einstein statistics of photons play an important role, so that we can observe bunching of photons in outputs of beam-splitters. Recent progress about KLM's linear optical quantum computing is reviewed in Ref. [11].

To implement the conditional sign-flip gate, KLM use the nonlinear sign-shift (NS) gate, which applies the following transformation to a superposition of the number states of photons $|n\rangle$ for $n = 0, 1, 2$: $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle - \gamma|2\rangle$ (with probability $1/4$). The NS gate is essential for KLM's scheme, and they construct this gate only from passive linear optics. Although KLM's NS gate is a nondeterministic gate that works with probability $1/4$, we can detect its false event by measuring an auxiliary photon.

In this paper, we discuss the method for implementing the NS gate with a one-dimensional Kerr-nonlinear photonic crystal.

A photonic crystal is a periodic structure of dielectric materials whose dielectric constants are different from each other. If a wavelength of a propagating electromagnetic wave is comparable with the period of the crystal, Bragg reflection occurs and the photonic band gap (the light dispersion relation) is formed [12, 13, 14]. This dispersion relation reduces the group velocity of the electromagnetic field of incident photons, and thus it enhances nonlinear interaction of the photons. If we construct the photonic crystal from alternating layers of a Kerr medium and a medium with linear polarization (a medium with $\chi^{(2)} = 0$ and $\chi^{(3)} \neq 0$ and a medium with $\chi^{(2)} = \chi^{(3)} = 0$, where $\chi^{(n)}$ is the n th-order nonlinear electric susceptibility) as a one-dimensional periodic structure, we can rotate the phase of the wave function of the incident photons by a large angle efficiently. (In general, for almost all the dielectric media, Kerr nonlinearity is too weak to rotate the phase of wave function of photons by a certain angle.) Using this method, we realize KLM's NS gate. In our implementation, the NS gate works with probability unity in principle.

Someone may make an objection to our proposal because we are going to introduce a nonlinear device into KLM's scheme that aims to construct a quantum computer only with passive linear optics. However, the author thinks that our proposal is a practical compromise because the Kerr medium is a well-studied material in the field of optics and

rapid progress has been made in experimental studies of photonic crystals. The idea of making use of the Kerr medium for the conditional sign-flip gate can be found in Ref. [15]. Inoue and Aoyagi show experimental evidence of enhancement of the nonlinearity in a Kerr-nonlinear photonic crystal in Ref. [16]. Numerical investigations of a one-dimensional Kerr-nonlinear photonic crystal are made in Refs. [17, 18]. Some research groups have recently proposed optical quantum computation schemes using weak cross-Kerr nonlinearities and strong probe coherent fields [19, 20, 21, 22, 23]. Their proposals and our method use optical nonlinearities in common with each other.

This paper is organized as follows: In the rest of this section, we give a brief review of KLM's scheme. In Sec. 2, we outline the NS gate realized by the Kerr-nonlinear photonic crystal. And then, we estimate the time-of-flight that the injected photons need for obtaining a certain angle of the phase rotation caused by nonlinear interaction in the homogeneous Kerr medium. In Sec. 3, we investigate the photonic band structure in the photonic crystal and evaluate the group velocity of the photons. From these results, we give a concrete example of a design for the photonic crystal that realizes the NS gate with giving numerical values of the period of the crystal, the thickness and the dielectric constant of each layer, and so on. In Sec. 4, we give discussions. In Appendix A, we explain the third-order nonlinear susceptibility $\chi^{(3)}$ and the nonlinear refraction coefficient n_2 . In Appendix B, we derive the wave equation and the effective Hamiltonian of the electromagnetic field in the nonlinear dielectric medium.

Here, we sketch out KLM's quantum circuit for implementing the conditional sign-flip gate. First, we construct a qubit from a pair of optical paths. The optical path (mode) forms a physical system which takes a superposition of the number state $|n\rangle$ for $n = 0, 1, 2, \dots$, where n is the number of photons on the path. $|0\rangle_{x1} \otimes |1\rangle_{x2}$ is a state where modes $x1$ and $x2$ have zero and one photons, respectively, and we regard it as a logical ket vector $|\bar{0}\rangle_x$. We regard $|1\rangle_{x1} \otimes |0\rangle_{x2}$ as a logical ket vector $|\bar{1}\rangle_x$, similarly. And we describe an arbitrary state of a qubit as $|\phi\rangle_x = \alpha|\bar{0}\rangle_x + \beta|\bar{1}\rangle_x$ for $|\alpha|^2 + |\beta|^2 = 1$. (This construction of a qubit is called the dual-rail qubit representation [15].)

Second, we define the NS operation as follows:

$$\begin{aligned} |\psi\rangle &= \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \\ &\rightarrow |\psi'\rangle = \alpha|0\rangle + \beta|1\rangle - \gamma|2\rangle. \end{aligned} \quad (1)$$

We pay attention to the fact that the NS operation does not work on a qubit but on a superposition of the number states of a single mode, $|0\rangle$, $|1\rangle$, and $|2\rangle$.

Third, we construct the conditional sign-flip gate from the NS operation as shown in Fig. 1. An optical network drawn in Fig. 1 works as the conditional sign-flip gate, whose operation is given by

$$\begin{aligned} |\bar{j}\rangle_x \otimes |\bar{k}\rangle_y &\rightarrow (-1)^{jk} |\bar{j}\rangle_x \otimes |\bar{k}\rangle_y \\ &\text{for } j, k \in \{0, 1\}. \end{aligned} \quad (2)$$

Let us confirm the function of this network. In Fig. 1, symbols $B1$ and $B2$ represent beamsplitters, which transform the incident number states of modes $a1$ and $a2$ as follows:

$$|n\rangle_{a1} |m\rangle_{a2} = \frac{1}{\sqrt{n!m!}} (a_1^\dagger)^n (a_2^\dagger)^m |0\rangle_{a1} |0\rangle_{a2}$$

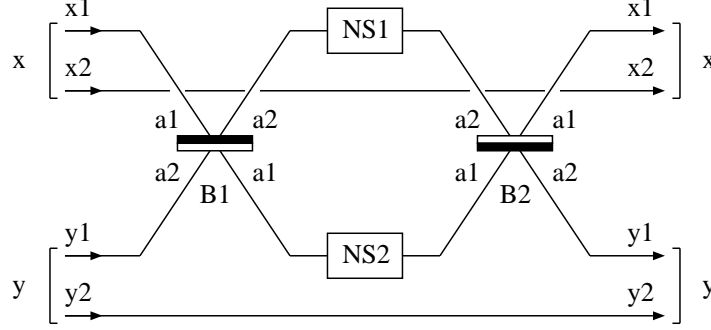


Figure 1: Implementation of the conditional sign-flip gate with the NS operations. Qubits x and y consist of a pair of modes $x1$ and $x2$ and a pair of modes $y1$ and $y2$, respectively. Symbols $B1$ and $B2$ represent beamsplitters. Symbols $NS1$ and $NS2$ represent the NS gates. Photons travel from left to right in this network.

$$\rightarrow \frac{1}{\sqrt{n!m!}} \left[\frac{1}{\sqrt{2}} (a_1^\dagger + a_2^\dagger) \right]^n \left[\frac{1}{\sqrt{2}} (a_1^\dagger - a_2^\dagger) \right]^m |0\rangle_{a1} |0\rangle_{a2} \quad (3)$$

for $n, m \in \{0, 1, 2, \dots\}$,

where a_1^\dagger and a_2^\dagger are creation operators of modes $a1$ and $a2$ respectively, and their commutation relations are given by $[a_j, a_k^\dagger] = \delta_{jk}$ and $[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0$ for $j, k \in \{1, 2\}$. The beamsplitters $B1$ and $B2$ replace a_1^\dagger and a_2^\dagger with $(1/\sqrt{2})(a_1^\dagger + a_2^\dagger)$ and $(1/\sqrt{2})(a_1^\dagger - a_2^\dagger)$, respectively. We give attention to the fact that $B2$ applies an inverse transformation of $B1$. Symbols $NS1$ and $NS2$ represent the NS gates that apply the operation given in Eq. (1) to the modes $a1$ and $a2$, respectively.

If we put a superposition of $|\bar{0}\rangle_x |\bar{0}\rangle_y$, $|\bar{0}\rangle_x |\bar{1}\rangle_y$, and $|\bar{1}\rangle_x |\bar{0}\rangle_y$ into the left side of the network shown in Fig. 1, the network leaves it untouched and returns it as an output from the right side of the network. However, if we put a state $|\bar{1}\rangle_x |\bar{1}\rangle_y = |1\rangle_{x1} |0\rangle_{x2} |1\rangle_{y1} |0\rangle_{y2}$ into the network, the following transformation is applied to the modes $x1$ and $y1$:

$$\begin{aligned} |1\rangle_{x1} |1\rangle_{y1} &\xrightarrow{B1} \frac{1}{\sqrt{2}} (|2\rangle_{a1} |0\rangle_{a2} - |0\rangle_{a1} |2\rangle_{a2}) \\ &\xrightarrow{NS1, NS2} -\frac{1}{\sqrt{2}} (|2\rangle_{a1} |0\rangle_{a2} - |0\rangle_{a1} |2\rangle_{a2}) \\ &\xrightarrow{B2} -|1\rangle_{x1} |1\rangle_{y1}. \end{aligned} \quad (4)$$

Thus, we obtain $-|\bar{1}\rangle_x |\bar{1}\rangle_y = -|1\rangle_{x1} |0\rangle_{x2} |1\rangle_{y1} |0\rangle_{y2}$ as an output for the input state $|\bar{1}\rangle_x |\bar{1}\rangle_y$.

Therefore, we confirm that the network shown in Fig. 1 realizes the conditional sign-flip gate. Moreover, we notice that implementing the NS gate is essential for the network in Fig. 1.

2 An outline of the NS gate

In this section, we explain the NS gate, which is constructed from a one-dimensional Kerr-nonlinear photonic crystal. As shown in Fig. 2, we inject photons whose wave

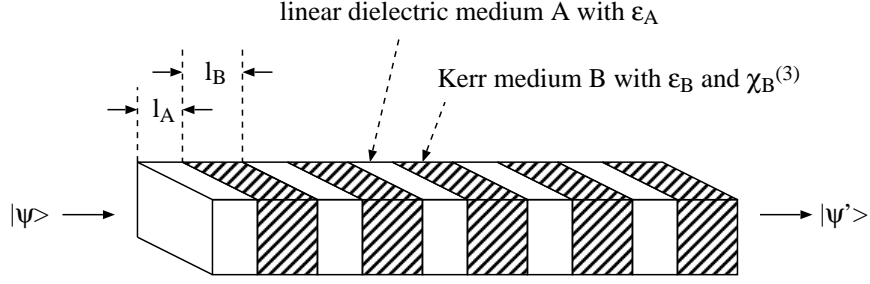


Figure 2: An outline of the NS gate constructed from a one-dimensional Kerr-nonlinear photonic crystal. The photonic crystal is composed of alternating layers of materials A and B with thicknesses l_A and l_B , respectively. [Thus, a period of the crystal is given by $(l_A + l_B)$.] The material A is a linear dielectric medium whose dielectric constant is given by ϵ_A , and the material B is a Kerr medium whose dielectric constant and third-order nonlinear susceptibility for optical Kerr effect are given by ϵ_B and $\chi_B^{(3)}$, respectively. A wave function of injected photons $|\psi\rangle$ propagates through the photonic crystal with nonlinear phase rotation of the optical Kerr effect enhanced by the slow group velocity, and it goes out of the photonic crystal as $|\psi'\rangle$.

function is given by $|\psi\rangle$ (a superposition of $|0\rangle$, $|1\rangle$, and $|2\rangle$) into the photonic crystal. The photonic crystal is composed of alternating layers of materials A and B , and they form a one-dimensional periodic structure with a period $(l_A + l_B)$. The material A is a linear dielectric medium whose dielectric constant is given by ϵ_A , and the material B is a Kerr medium whose dielectric constant and third-order nonlinear susceptibility for optical Kerr effect are given by ϵ_B and $\chi_B^{(3)}$, respectively. While the wave function of the photons travels through the photonic crystal, it undergoes Bragg reflection and its group velocity decreases. At the same time, the phase of the wave function is rotated by the optical Kerr effect induced by the layers of the Kerr medium B . Because the slow group velocity enhances the Kerr nonlinearity, we can expect the phase to rotate by a large angle efficiently.

Here, we estimate the rotational angle of the phase caused by the optical Kerr effect in a homogeneous Kerr medium by way of trial. (We do not investigate the phase rotation in a one-dimensional Kerr-nonlinear photonic crystal here. We evaluate the phase rotation caused by the photonic crystal in Sec. 3.) According to the Refs. [24, 25], we have the effective Hamiltonian of photons in the homogeneous Kerr medium with ϵ and $\chi^{(3)}$ as

$$\mathcal{H} \simeq \hbar\omega(a^\dagger a + \frac{1}{2}) + \hbar\chi(a^\dagger)^2 a^2, \quad (5)$$

and

$$\chi = \frac{9}{8} \frac{\hbar\omega^2 \chi^{(3)}}{\epsilon^2 S d}, \quad (6)$$

where ω is a circular frequency of the photons, S represents the cross-sectional area of a device made of the Kerr medium, and d represents the width of the wave-packet of the photons in the medium. Thus, Sd is the volume for the quantization of photons. [While

$\chi^{(3)}$ is given by a fourth-rank tensor as $\mathbf{P} = \chi^{(1)} : \mathbf{E} + \chi^{(3)} : \mathbf{EEE}$ in general, $\chi^{(3)}$ in Eq. (6) represents a certain component of the fourth-rank tensor. In Appendix A, we give a brief explanation about $\chi^{(3)}$ and the nonlinear refraction coefficient n_2 , which is observed in an experiment direct. In Appendix B, we show derivation of the effective Hamiltonian given by Eqs. (5) and (6).]

To derive the effective Hamiltonian given by Eqs. (5) and (6), we carry out the following process. First of all, we perform the quantization of the free electromagnetic field whose Hamiltonian does not include $\chi^{(3)}$. Next, regarding $\chi^{(3)}$ as an expansion parameter, we evaluate the first-order perturbed energy, which we can consider to be the effective Hamiltonian. (This process is explained in Appendix B.) In the derivation of the Hamiltonian given in Eqs. (5) and (6), we assume that photons travel in the homogeneous Kerr medium. However, the system that we discuss in this paper is photons propagating through the photonic crystal. Thus, we have to consider the wave function of photons which extends over many layers of the photonic crystal. Therefore, the Hamiltonian given by Eqs. (5) and (6) offers us a rough picture of the phase rotation. (Moreover, although we discuss the phase rotation by quantum mechanics in this section, we derive the photonic band gap by classical theory of electrodynamics in Sec. 3. Hence, our analysis of the quantum gate is semi-classical.)

According to Eq. (5), the number state of the photons $|n\rangle$ for $n = 0, 1, 2, \dots$ obtains the phase rotation $\exp[i\chi n(n-1)t]$, where t is photons' time-of-flight in the crystal. [We remember $(a^\dagger)^2 a^2 |n\rangle = n(n-1)|n\rangle$. Because the first term of Eq. (5) (namely, the unperturbed Hamiltonian) gives only the background phase rotation $\exp\{i\omega[n + (1/2)]t\}$ for $|n\rangle$, we can neglect it.] Thus, the Kerr medium transforms the wave function of the traveling photons as $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle + e^{i2\chi t}\gamma|2\rangle$. If we let $|\chi t|$ be equal to $\pi/2$, we can use this photonic crystal as the NS gate.

In Refs. [26, 27, 28], nonlinear optical properties of GaAs/GaAlAs multiple quantum well (MQW) material are studied. In Ref. [27], Miller *et al.* prepare a sample that consists of 84 periods of 144-Å GaAs and 102-Å GaAs/Ga_{0.7}Al_{0.3}As and obtain the nonlinear refraction coefficient $n_2 \sim 1.2 \times 10^{-4}$ [cm²/W] at wavelength (~ 850 nm). The dielectric constants of GaAs and GaAlAs are given by $\epsilon = 13.0 \times \epsilon_0$, where $\epsilon_0 (= 8.85 \times 10^{-12}$ [F/m]) is the dielectric constant of vacuum. From the result of $n_2 \sim 1.2 \times 10^{-4}$ [cm²/W], the third-order nonlinear susceptibility of GaAs/GaAlAs MQW structure is given by $\chi^{(3)} = 2.97 \times 10^{-2}$ (esu) = 3.67×10^{-21} [m³·C/V³] (SI unit). This $|\chi^{(3)}|$ is much larger than those of other materials. [The relation between n_2 and $\chi^{(3)}$ is explained in Appendix A [29].]

Let us estimate $t = \pi/(2\chi)$, the time-of-flight for realizing the conditional sign-flip operation, numerically. We assume that the wavelength of the incident photons is given by $\lambda = 8.47 \times 10^{-7}$ [m] [$\omega = 2\pi(c/\lambda) = 2.23 \times 10^{15}$ [rad/s], $\nu = c/\lambda = 3.54 \times 10^{14}$ [1/s]] which is slightly out of tune from 850 nm. (We use $c = 3.00 \times 10^8$ [m/s].) We also assume that the incident photons form a wave-packet whose cross-sectional area and width in vacuum are given by $S = 1.0 \times 10^{-8}$ [m²] (a square with the side-length 0.1 mm) and $d_0 = 4.24 \times 10^{-6}$ [m] (five times as long as the wavelength), respectively. This wave-packet corresponds with a 14.1 femto-second pulse. Because the width of the wave-packet is shortened in the medium as $d = \sqrt{(\epsilon_0\mu_0)/(\epsilon\mu)}d_0$, where μ_0 and μ represent the magnetic permeabilities of vacuum and the material respectively, we obtain $d = 1.18 \times 10^{-6}$ [m] in

GaAs/GaAlAs. (We use the fact $\mu \simeq \mu_0$ for GaAs and GaAlAs.) The Planck constant is given by $\hbar = h/(2\pi) = 1.05 \times 10^{-34}$ [J·s]. Substituting the above numerical values into Eq. (6), we obtain $\chi = 1.38 \times 10^{10}$ [1/s]. Thus, we need $t = \pi/(2\chi) = 1.14 \times 10^{-10}$ [s] for the time-of-flight in GaAs/GaAlAs MQW material. To obtain the time-of-flight $t = 1.14 \times 10^{-10}$ [s] in the GaAs/GaAlAs MQW material, we have to prepare a device whose length is equal to $L = \sqrt{(\epsilon_0\mu_0)/(\epsilon\mu)}ct = 9.49 \times 10^{-3}$ [m]. (In this evaluation, we assume that the velocity of the photons in the Kerr medium is not affected by the term of χ . This is because the contribution of the term of χ is much smaller than the Hamiltonian of the free photons. We notice that the term of χ in the Hamiltonian increases in proportion to the square of the number of the photons and we consider the case where the number of the photons is equal to two at most now. Details of this discussion is given in Sec. 3.)

In the above estimation, the volume for the quantization of photons Sd is crucial. We assume $S = 1.0 \times 10^{-8}$ [m²] and this quantity corresponds with the cross-sectional area of the device of the GaAs/GaAlAs MQW material. To prevent the wave function of photons from tunneling outside of the box for quantization, we have to wrap up the device of GaAs/GaAlAs in a material whose refractive index is larger than refractive indexes of both GaAs and GaAlAs. This means that we have to make a cavity of the GaAs/GaAlAs MQW material for confining photons inside, and the cross-sectional area and the length of the cavity is given by $S = 1.0 \times 10^{-8}$ [m²] and $L = 9.49 \times 10^{-3}$ [m], respectively.

The author cannot judge whether this requirement is feasible for realizing in the laboratory. Because the ratio between two side-lengths of the cavity is very large, it seems to be difficult to fabricate this device. To overcome this problem, we reduce the group velocity of photons in the Kerr medium by the dispersion relation induced by the photonic crystal.

3 Dispersion relation of the Kerr-nonlinear photonic crystal

In this section, we investigate the photonic band gap (the dispersion relation) induced in the photonic crystal of Fig. 2. And then, we derive the group velocity of incident photons from this dispersion relation.

The wave equation of the electromagnetic field in a one-dimensional Kerr-nonlinear photonic crystal is given by

$$\begin{aligned} \frac{\partial^2}{\partial z^2} E(z, t) - \mu_0 \epsilon(z) \frac{\partial^2}{\partial t^2} E(z, t) \\ - \mu_0 \chi^{(3)}(z) \frac{\partial^2}{\partial t^2} E(z, t)^3 = 0, \end{aligned} \quad (7)$$

$$\frac{\partial}{\partial z} E(z, t) = -\frac{\partial}{\partial t} B(z, t), \quad (8)$$

where

$$\epsilon(z + l_A + l_B) = \epsilon(z) \quad \text{for } -\infty < x < \infty, \quad (9)$$

$$\epsilon(z) = \begin{cases} \epsilon_A(> 0) & 0 \leq z < l_A \\ \epsilon_B(> 0) & l_A \leq z < l_A + l_B \end{cases}, \quad (10)$$

$$\chi^{(3)}(z + l_A + l_B) = \chi^{(3)}(z) \quad \text{for } -\infty < x < \infty, \quad (11)$$

$$\chi^{(3)}(z) = \begin{cases} 0 & 0 \leq z < l_A \\ \chi^{(3)} & l_A \leq z < l_A + l_B \end{cases}. \quad (12)$$

In Eq. (7), we assume that the photonic crystal is periodic in the z -direction and uniform in the x - y plane, and the electromagnetic wave propagates in the positive z -direction. Derivation of Eqs. (7) and (8) is shown in Appendix B.

Here, we assume that the medium B is the GaAs/GaAlAs MQW material and all physical quantities are given in Sec. 2. We can estimate the contribution made by the term $\mu_0 \chi^{(3)}(z)(\partial^2/\partial t^2)E(z, t)^3$ of Eq. (7) as follows: Because of Eq. (6) and Eqs. (69), (71), (76) in Appendix B, we obtain the relation

$$\chi^{(3)}E^2 \sim \chi^{(3)} \frac{\hbar\omega}{\epsilon S d} = \epsilon \frac{8}{9} \frac{\chi}{\omega}. \quad (13)$$

(We pay attention to the fact that the number of photons in the wave-packet is equal to two at most.) And, using $\chi = 1.38 \times 10^{10}$ [1/s] and $\omega = 2.23 \times 10^{15}$ [rad/s], we obtain

$$\frac{8}{9} \frac{\chi}{\omega} = 5.50 \times 10^{-6}. \quad (14)$$

Thus, we arrive at

$$\chi^{(3)}E^2 \ll \epsilon. \quad (15)$$

Hence, we can neglect the term that include $\chi^{(3)}$ in Eq. (7) for evaluating the photonic band gap. Moreover, we take monochromatic light as an approximation of the injected pulse into the photonic crystal, so that we only need to consider a stationary solution,

$$E(z, t) = E(z)e^{-i\omega t}. \quad (16)$$

(In Sec. 2, we assume that the signal injected to the photonic crystal is a femto-second pulse of the typical length 5λ . Propagation of ultrashort pulses in the Kerr-nonlinear photonic crystal is considered in Ref. [17]. Scalora *et al.* indicate that the nonlinearity causes various effects to the pulse near the band edge. However, because we need hard calculations for analyzing the behavior of the ultrashort pulse in the photonic crystal and it is beyond the purpose of this paper, we neglect these effects for simplicity. Thus, we adopt the approximation by the monochromatic light.) Thus, the wave equation that we have to solve is written down as

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\epsilon(z)}{\epsilon_0} \left(\frac{\omega}{c} \right)^2 \right] E(z) = 0, \quad (17)$$

where Eqs. (9) and (10) are assumed. This problem is known as Kronig-Penney model and we can derive its exact solutions [30, 31].

Let us solve Eqs. (9), (10), and (17). Solutions of the region I ($0 < z < l_A$) and the region II ($l_A < z < l_A + l_B$) for Eqs. (10) and (17) are given by

$$E_j(z) = C_{j+} \exp(iK_j z) + C_{j-} \exp(-iK_j z) \quad \text{for } j = \text{I, II}, \quad (18)$$

where

$$K_{\text{I}} = \frac{\omega}{c} \sqrt{\frac{\epsilon_A}{\epsilon_0}}, \quad K_{\text{II}} = \frac{\omega}{c} \sqrt{\frac{\epsilon_B}{\epsilon_0}}. \quad (19)$$

At points where a finite potential step exists, the above solution must be continuous together with their derivatives (that is, the continuity conditions). Moreover, because $\epsilon(z)$ is periodic, we can apply Bloch's theorem to the continuity conditions of the solution.

Bloch's Theorem: Suppose $E_k(z)$ is a solution of Eqs. (9) and (17). Then, $E_k(z)$ satisfies the following relations.

$$E_k(z) = e^{ikz} u_k(z), \quad (20)$$

$$u_k(z + l_A + l_B) = u_k(z) \quad \text{for } -\infty < z < \infty, \quad (21)$$

$$-\frac{\pi}{l_A + l_B} \leq k \leq \frac{\pi}{l_A + l_B}. \quad (22)$$

Thus, we can require the continuity conditions to $E_{\text{I}}(z)$ and $E_{\text{II}}(z)$ as follows:

$$E_{\text{I}}(0) = E_{\text{II}}(0), \quad (23)$$

$$\left. \frac{d}{dz} E_{\text{I}} \right|_{z=0} = \left. \frac{d}{dz} E_{\text{II}} \right|_{z=0}, \quad (24)$$

$$E_{\text{I}}(l_A) = e^{ik(l_A + l_B)} E_{\text{II}}(-l_B), \quad (25)$$

$$\left. \frac{d}{dz} E_{\text{I}} \right|_{z=l_A} = e^{ik(l_A + l_B)} \left. \frac{d}{dz} E_{\text{II}} \right|_{z=-l_B}. \quad (26)$$

The above four equations can be rewritten in the form

$$M \begin{pmatrix} C_{\text{I}+} \\ C_{\text{I}-} \\ C_{\text{II}+} \\ C_{\text{II}-} \end{pmatrix} = 0, \quad (27)$$

where

$$M = \begin{pmatrix} 1 & 1 & -1 & -1 \\ K_{\text{I}} & -K_{\text{I}} & -K_{\text{II}} & K_{\text{II}} \\ P & 1/P & -Q/R & -QR \\ K_{\text{I}}P & -K_{\text{I}}/P & -K_{\text{II}}Q/R & K_{\text{II}}QR \end{pmatrix}, \quad (28)$$

$$P = \exp(iK_{\text{I}}l_A), \quad (29)$$

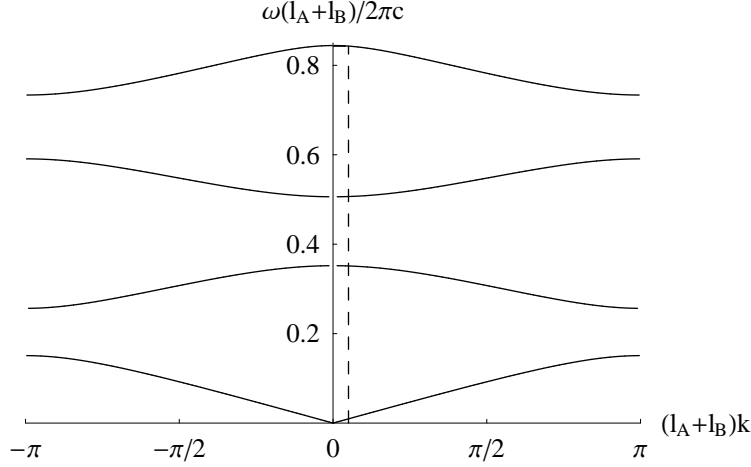


Figure 3: The dispersion relation of the photonic crystal with $l_A = l_B = 3.57 \times 10^{-7}$ [m], $\epsilon_A/\epsilon_0 = 1.0$, and $\epsilon_B/\epsilon_0 = 13.0$. Broken lines indicates a point where $\omega(l_A + l_B)/2\pi c = 0.843$ (that is, $\omega = 2.23 \times 10^{15}$ [rad/s] and $\lambda = 8.47 \times 10^{-7}$ [m]) and $(l_A + l_B)k = 0.158$. This point belongs to the fourth conduction band.

$$Q = \exp[ik(l_A + l_B)], \quad (30)$$

$$R = \exp(iK_{II}l_B). \quad (31)$$

From Eq. (27), we obtain $\det M = 0$, and it implies

$$\begin{aligned} & \cos[(l_A + l_B)k] - \cos(l_A K_I) \cos(l_B K_{II}) \\ & + \frac{K_I^2 + K_{II}^2}{2K_I K_{II}} \sin(l_A K_I) \sin(l_B K_{II}) = 0. \end{aligned} \quad (32)$$

From Eq. (32), we can obtain the light dispersion relation $\omega = \omega(k)$.

Here, we substitute physical quantities introduced in Sec. 2 into Eq. (32). The implementation is as follows: The materials *A* and *B* are air and GaAs/GaAlAs MQW structure, respectively. The dielectric constants of them are given by $\epsilon_A/\epsilon_0 = 1.0$ and $\epsilon_B/\epsilon_0 = 13.0$. We assume the thicknesses of layers that compose the photonic crystal are given by $l_A = l_B = 3.57 \times 10^{-7}$ [m]. Using Eqs. (19) and (32), we obtain the light dispersion relation drawn in Fig. 3. The group velocity of the light in the photonic crystal can be written in the form

$$v_g(k_0) = \left. \frac{d\omega}{dk} \right|_{k_0}. \quad (33)$$

We draw the group velocity $v_g(k)$ of the fourth conduction band in Fig. 4.

Because the variation of v_g against k is very steep in the region where v_g takes a small value as shown in Fig. 4, we need to tune up the wavelength of injected photons precisely for obtaining the slow group velocity. (At the same time, we have to adjust l_A and l_B precisely because v_g is sensitive to the thickness of the layers, as well.)

If we inject photons with $\lambda = 8.5 \times 10^{-7}$ [m] and $\omega = 2.22 \times 10^{15}$ [rad/s] into the photonic crystal, its wave vector and group velocity are given by $(l_A + l_B)k = 0.295$ and

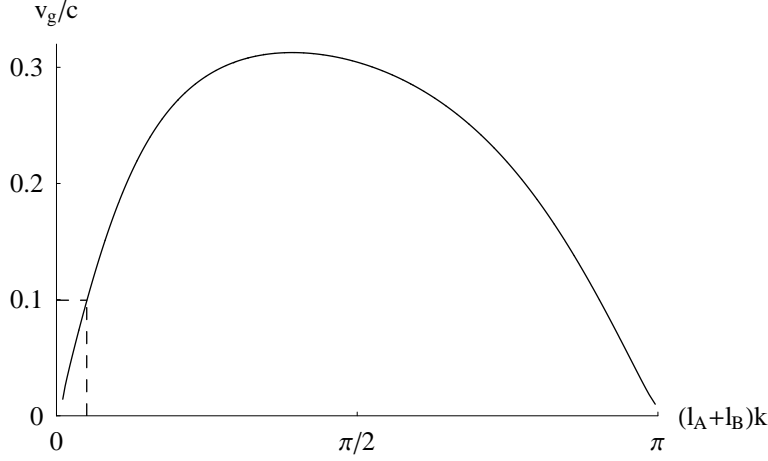


Figure 4: The group velocity $v_g(k)/c$ of the fourth conduction band in Fig. 3. When we have $(l_A + l_B)k = 0.158$, we obtain $v_g/c = 0.0995$. Broken lines indicates this point.

$v_g/c = 0.171$, respectively. On the other hand, if we inject photons with $\lambda = 8.47 \times 10^{-7}$ [m] and $\omega = 2.23 \times 10^{15}$ [rad/s] into the photonic crystal [this corresponds with $\omega(l_A + l_B)/2\pi c = 0.843$], its wave vector and group velocity are given by $(l_A + l_B)k = 0.158$ and $v_g/c = 0.0995$, respectively.

Now, we obtain the group velocity v_g of photons in the photonic crystal. However, photons injected into the photonic crystal undergo Kerr nonlinear interaction only while they travel through the layers of the material B . Thus, we have to estimate the probability P_A that the photons stay in the layers of the material A and the probability $P_B (= 1 - P_A)$ that the photons stay in the layers of the material B . Because the number of photons in a certain box is in proportion to energy contained in it, we obtain the relation

$$\begin{aligned}
 P_A : P_B &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt S \int_0^{l_A} dz \frac{1}{2} [\epsilon_A E_I(z, t)^2 + \frac{1}{\mu_0} B_I(z, t)^2] : \\
 &\quad \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt S \int_{l_A}^{l_A+l_B} dz \frac{1}{2} [\epsilon_B E_{II}(z, t)^2 + \frac{1}{\mu_0} B_{II}(z, t)^2] \\
 &= \int_0^{2\pi/\omega} dt \int_0^{l_A} dz [\frac{\epsilon_A}{\epsilon_0 c^2} E_I(z, t)^2 + B_I(z, t)^2] : \\
 &\quad \int_0^{2\pi/\omega} dt \int_{l_A}^{l_A+l_B} dz [\frac{\epsilon_B}{\epsilon_0 c^2} E_{II}(z, t)^2 + B_{II}(z, t)^2], \tag{34}
 \end{aligned}$$

where $E_j(z, t)$ and $B_j(z, t)$ are given by

$$\begin{aligned}
 E_j(z, t) &= \text{Re}\{[C_{j+} \exp(iK_j z) + C_{j-} \exp(-iK_j z)] \exp(-i\omega t)\}, \\
 B_j(z, t) &= \text{Re}\{\frac{K_j}{\omega} [C_{j+} \exp(iK_j z) - C_{j-} \exp(-iK_j z)] \exp(-i\omega t)\} \\
 &\quad \text{for } j = \text{I, II}. \tag{35}
 \end{aligned}$$

[We take an average of the energy over time t in Eq. (34).]

Substituting $\omega(l_A + l_B)/2\pi c = 0.843$ that corresponds with $\lambda = 8.47 \times 10^{-7}$ [m] and $(l_A + l_B)k = 0.158$ into the matrix M defined in Eq. (28), we can obtain the coefficients of the electromagnetic field, C_{I+} , C_{I-} , C_{II+} , and C_{II-} from Eq. (27). After carrying out slightly tough numerical calculations, we obtain

$$P_A : P_B \simeq 1 : 11.2. \quad (36)$$

In Sec. 2, we obtain the time-of-flight $\tau_{\text{tof}} = \pi/(2\chi) = 1.14 \times 10^{-10}$ [s] that we need to flip the sign of the phase in the homogeneous GaAs/GaAlAs MQW material. Thus, the length of the photonic crystal for realizing the NS gate is given by

$$L_{\text{PhC}} = \tau_{\text{tof}} \frac{P_A + P_B}{P_B} v_g = 3.71 \times 10^{-3} \text{ [m]}. \quad (37)$$

The photonic crystal with the length $L_{\text{PhC}} = 3.71 \times 10^{-3}$ [m] consists of about 10 400 layers (about 5200 periods) of materials A and B (the air and the GaAs/GaAlAs MQW structure). Moreover, each layer of the GaAs/GaAlAs MQW structure consists of about 15 periods of GaAs and GaAs/GaAlAs (the width of each quantum well is equal to about 100-Å).

In the above estimation, we cannot obtain a great reduction in the length of the device by the slow group velocity in the photonic crystal. (The lengths of the homogeneous sample of the Kerr medium and the sample of the Kerr-nonlinear photonic crystal for the phase rotation are equal to 9.49 [mm] and 3.71 [mm], respectively.) To obtain a significant reduction, we have to adjust the wavelength of the injected photons more precisely. However, as shown in Fig. 4, the variation of the group velocity v_g against the wave vector k is very steep when v_g takes a small value, so that an accurate adjustment of the wavelength of the injected photons is very difficult. This is a weak point of our proposition. However, the author thinks that we can expect the great reduction in the length of the device by a precise tuning up of the parameters.

4 Discussion

According to our estimation, to realize the NS gate, we have to construct the photonic crystal from about ten thousands layers, each of whose thickness is equal to about $0.4 \mu\text{m}$. Moreover, the layers of the Kerr medium has the GaAs/GaAlAs MQW structure. These requirements seem to be very severe. However, for example, Noda and his collaborators fabricate woodpile-structure of GaAs, whose typical period is equal to $0.7 \mu\text{m}$ [32, 33]. Hence, the author believes that we can construct the NS gate from the one-dimensional Kerr-nonlinear photonic crystal in the not-too-distant future.

The variation of the group velocity v_g against the wave vector k is very steep in the region where v_g takes a small value as shown in Fig. 4. Thus, v_g is sensitive to the thickness of the layers that compose the photonic crystal and the wavelength of injected photons, so that we have to adjust them precisely. In this paper, we show a plan for the NS gate with the GaAs/GaAlAs MQW structure as a concrete example. In our example, we cannot obtain a significant reduction in the length of the device caused by the slow group velocity in the photonic crystal, compared to the homogeneous sample of the Kerr

medium. However, the author believes that careful tuning up of the parameters in the laboratory realizes a feasible design of the device. Because the adjustment of the physical quantities is very subtle, we may find another design that is better than the plan we have shown in this paper.

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A The third-order nonlinear susceptibility and the nonlinear refraction coefficient

In this section, we explain the relation between $\chi^{(3)}$ (the third-order nonlinear susceptibility) and n_2 (the nonlinear refraction coefficient), which is obtained by an experiment for a Kerr medium direct. (Details of this topic can be found in Ref. [29].) Using this relation, we can calculate $\chi^{(3)}$ from an experimental data of n_2 .

In general, a nonlinear dielectric polarization of a material is given by a function of the electric field as

$$P_i = \sum_j \chi_{ij}^{(1)} E_j + \sum_{jk} \chi_{ijk}^{(2)} E_j E_k + \sum_{jkl} \chi_{ijkl}^{(3)} E_j E_k E_l + \dots$$

for $i, j, k, l \in \{x, y, z\}$,

(38)

where $\chi^{(n)}$ represents the tensor of the n th-order nonlinear susceptibility. Let us assume that an electromagnetic plane wave propagates in the positive z -direction in the material, so that the electric field \mathbf{E} and the magnetic flux density \mathbf{B} lie in the x - y plane. Letting the x axis and the y -axis be parallel to \mathbf{E} and \mathbf{B} respectively, we can write the electric field as $\mathbf{E} = (E, 0, 0)$. Moreover, to simplify \mathbf{P} defined in Eq. (38), we make the following assumptions about $\chi^{(n)}$:

$$\begin{aligned} \chi_{ij}^{(1)} &= \delta_{ij} \chi^{(1)} \\ \chi_{ijk}^{(2)} &= 0 \quad \text{for } \forall i, j, k \\ \chi_{ijkl}^{(3)} &= \begin{cases} \chi_i^{(3)} & i = j = k = l \\ 0 & \text{others} \end{cases}, \end{aligned}$$
(39)

and $\chi_{ij\dots}^{(n)} = 0$ for $n \geq 4$. Moreover, we write $\chi_x^{(3)} = \chi^{(3)}$.

From the above assumptions, we obtain the dielectric polarization $\mathbf{P} = (P, 0, 0)$, where

$$P = \chi^{(1)} E + \chi^{(3)} E^3$$
(40)

Thus, the electric flux density is given by $\mathbf{D} = (D, 0, 0)$ and

$$D = \epsilon_0 E + P = (\epsilon_0 + \chi^{(1)}) E + \chi^{(3)} E^3,$$
(41)

where ϵ_0 is the dielectric constant of vacuum. We define the nonlinear dielectric constant of the material as

$$\epsilon(E) = \frac{D}{E} = \epsilon_0 + \chi^{(1)} + \chi^{(3)} E^2.$$
(42)

[In general, we regard $\epsilon(0)[= \epsilon_0 + \chi^{(1)}]$ as the dielectric constant of the material.]

The refractive index of the matter is given by

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}, \quad (43)$$

where μ_0 and μ are the magnetic permeabilities of vacuum and the material, respectively.

Assuming $|\epsilon_0 + \chi^{(1)}| \gg |\chi^{(3)}E^2|$ and $\mu \simeq \mu_0$, which is satisfied for almost all the materials, we can expand n of Eq. (43) in powers of $\chi^{(3)}$ as

$$n = n_0 + n'_2 E^2 + \dots, \quad (44)$$

where

$$\begin{aligned} n_0 &= \sqrt{1 + \frac{\chi^{(1)}}{\epsilon_0}}, \\ n'_2 &= \frac{\chi^{(3)}}{2n_0\epsilon_0}. \end{aligned} \quad (45)$$

However, because the nonlinear refraction index n is obtained as a function of the field intensity $I = (1/2)\sqrt{\epsilon/\mu}E^2$ in the experiment, we rewrite Eq. (44) in the form of

$$n = n_0 + n_2 I, \quad (46)$$

where

$$n_2 = \frac{\chi^{(3)}}{n_0\epsilon_0} \sqrt{\frac{\mu}{\epsilon}} \simeq \frac{\chi^{(3)}}{n_0^2\epsilon_0^2 c}, \quad (47)$$

and $c = 1/\sqrt{\mu_0\epsilon_0}$ is the light velocity in vacuum.

In Refs. [26, 27, 28], nonlinear optical properties of GaAs/GaAlAs multiple quantum well (MQW) material are studied. In Ref. [27], Miller *et al.* prepare a sample that consists of 84 periods of 144-Å GaAs and 102-Å GaAs/Ga_{0.7}Al_{0.3}As and obtain the nonlinear refraction coefficient $n_2 \sim 1.2 \times 10^{-4}$ [cm²/W] at wavelength (~ 850 nm). Substituting this result of the experiment into Eq. (47), we obtain $\chi^{(3)} = 2.97 \times 10^{-2}$ (esu) = 3.67×10^{-21} [m·C/V³] (SI unit). (The value of $\chi^{(3)}$ is often represented in cgs-esu unit. We have the convenient relation, $\chi_{\text{esu}}^{(3)} = 8.1 \times 10^{18} \times \chi_{\text{SI}}^{(3)}$.) In the above calculations, we use the following facts: The dielectric constant of vacuum is given by $\epsilon_0 = 8.85 \times 10^{-12}$ [F/m]. Both the dielectric constants of GaAs and GaAlAs are given by $\epsilon = 13.0 \times \epsilon_0$. The magnetic permeabilities of GaAs and GaAlAs are given by $\mu \simeq \mu_0$, where μ_0 is the magnetic permeability of vacuum. Thus, we obtain $n_0 \simeq \sqrt{13.0}$. The light velocity in vacuum is given by $c = 3.00 \times 10^8$ [m/s].

The value of $|\chi^{(3)}|$ for GaAs/GaAlAs MQW structure given above is much larger than those of other materials. [For example, $\chi^{(3)}$ of CS₂, which is a typical Kerr medium, is given by 3.9×10^{-13} (esu).]

B The wave equation and the effective Hamiltonian of the electromagnetic field in the nonlinear dielectric medium

In this section, we derive the wave equation and the effective Hamiltonian of the electromagnetic field in the nonlinear dielectric medium.

First, we consider the wave equation of the electromagnetic field in the nonlinear dielectric medium. We use this equation for deriving the photonic band gap in Sec. 3. We start from Maxwell's equations in a matter, where there are no currents and no charges,

$$\nabla \cdot \mathbf{D}(\mathbf{x}, t) = 0, \quad (48)$$

$$\nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0, \quad (49)$$

$$\nabla \times \mathbf{H}(\mathbf{x}, t) - \frac{\partial}{\partial t} \mathbf{D}(\mathbf{x}, t) = 0, \quad (50)$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) + \frac{\partial}{\partial t} \mathbf{B}(\mathbf{x}, t) = 0. \quad (51)$$

In these four equations, \mathbf{D} and \mathbf{B} represent the electric and the magnetic flux densities, and \mathbf{E} and \mathbf{H} represent the electric and the magnetic fields, respectively. Moreover, we assume the following relations:

$$\begin{aligned} \mathbf{D}(\mathbf{x}, t) &= \epsilon_0 \mathbf{E}(\mathbf{x}, t) + \mathbf{P}(\mathbf{x}, t), \\ \mathbf{B}(\mathbf{x}, t) &= \mu(\mathbf{x}) \mathbf{H}(\mathbf{x}, t), \end{aligned} \quad (52)$$

where \mathbf{P} represents the dielectric polarization, ϵ_0 represents the dielectric constant of vacuum, and μ represents the magnetic permeability of the material. Here, we assume $\mu(\mathbf{x}) = \mu_0$, where μ_0 is the magnetic permeability of vacuum, because almost all the materials satisfy it. We also assume that the matter is uniform in the x - y plane.

Let us consider the electromagnetic wave that propagates in the positive z -direction. We can describe \mathbf{E} and \mathbf{B} as functions of z and t . Because the matter is uniform in the x - y plane, we can describe \mathbf{P} as a function of z and t , as well.

Using Eqs. (48), (49), (50), (51) and the above assumptions, we can obtain the following relations:

$$\frac{\partial}{\partial z}(\epsilon_0 E_z + P_z) = 0, \quad (53)$$

$$\frac{\partial}{\partial z} B_z = 0, \quad (54)$$

$$\frac{\partial}{\partial z} B_y + \mu_0 \frac{\partial}{\partial t}(\epsilon_0 E_x + P_x) = 0, \quad (55)$$

$$\frac{\partial}{\partial z} B_x - \mu_0 \frac{\partial}{\partial t}(\epsilon_0 E_y + P_y) = 0, \quad (56)$$

$$\frac{\partial}{\partial t}(\epsilon_0 E_z + P_z) = 0, \quad (57)$$

$$\frac{\partial}{\partial z} E_y - \frac{\partial}{\partial t} B_x = 0, \quad (58)$$

$$\frac{\partial}{\partial z}E_x + \frac{\partial}{\partial t}B_y = 0, \quad (59)$$

$$\frac{\partial}{\partial t}B_z = 0. \quad (60)$$

From Eqs. (53) and (57), we can take

$$\epsilon_0 E_z(z, t) + P_z(z, t) = 0. \quad (61)$$

In a similar way, from Eqs. (54) and (60), we can take

$$B_z(z, t) = 0. \quad (62)$$

Here, we assume that the material is a Kerr medium whose nonlinear susceptibilities are given by Eq. (39). Thus, the dielectric polarization is given by

$$P_i = \chi^{(1)}(z)E_i + \chi_i^{(3)}(z)E_i^3, \quad (63)$$

(we assume that $\chi^{(1)}$ and $\chi_i^{(3)}$ do not depend on t), and we obtain

$$E_z(z, t) = 0. \quad (64)$$

Now, $\mathbf{E}(z, t)$ and $\mathbf{B}(z, t)$ lie in the x - y plane. Letting the x -axis be parallel with $\mathbf{E}(z, t)$, we can write $\mathbf{E}(z, t) = (E(z, t), 0, 0)$. Thus, from Eqs. (56) and (58), we obtain $\mathbf{B}(z, t) = (0, B(z, t), 0)$. From Eqs. (55) and (59), we obtain the wave equations for $E(z, t)$ and $B(z, t)$,

$$\begin{aligned} \frac{\partial^2}{\partial z^2}E(z, t) - \mu_0\epsilon(z)\frac{\partial^2}{\partial t^2}E(z, t) \\ - \mu_0\chi^{(3)}(z)\frac{\partial^2}{\partial t^2}E(z, t)^3 = 0, \end{aligned} \quad (65)$$

$$\frac{\partial}{\partial z}E(z, t) = -\frac{\partial}{\partial t}B(z, t), \quad (66)$$

where $\epsilon(z) = \epsilon_0 + \chi^{(1)}(z)$ and $\chi^{(3)} = \chi_x^{(3)}$.

Second, we consider the effective Hamiltonian of the electromagnetic field in a homogeneous Kerr medium. We assume that the electromagnetic field propagates in the positive z -direction, and it is described by Eqs. (65) and (66). Because the material is uniform in all directions, we can regard $\chi^{(1)}$, $\chi^{(3)}$, and ϵ as constants.

In general, the energy of the electromagnetic field confined in the box of volume Vol is given by

$$\mathcal{H} = \int_{\text{Vol}} d^3\mathbf{x} \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B}). \quad (67)$$

Because we make the following assumptions now, $\mathbf{E} = (E(z, t), 0, 0)$, $\mathbf{D} = (\epsilon E(z, t) + \chi^{(3)}E(z, t)^3, 0, 0)$, $\mathbf{B} = (0, B(z, t), 0)$, $\mathbf{H} = (0, (1/\mu_0)B(z, t), 0)$, we can rewrite \mathcal{H} given in Eq. (67) as

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + V, \\ \mathcal{H}_0 &= \int_{\text{Vol}} d^3\mathbf{x} \frac{1}{2}(\epsilon E^2 + \frac{1}{\mu_0}B^2), \\ V &= \int_{\text{Vol}} d^3\mathbf{x} \frac{\chi^{(3)}}{2}E^4. \end{aligned} \quad (68)$$

From now on, we perform the quantization of the free electromagnetic field whose Hamiltonian is given by \mathcal{H}_0 . We regard V as a perturbation, and we construct the effective Hamiltonian of the quantized field by evaluating the first-order perturbed energy. For the quantization of the free field, we write $E(z, t)$ and $B(z, t)$ as products:

$$E(z, t) = -\frac{1}{\sqrt{\epsilon}} \sum_l \dot{q}_l(t) u_l(z), \quad (69)$$

$$B(z, t) = \frac{1}{\sqrt{\epsilon}} \sum_l q_l(t) u'_l(z). \quad (70)$$

The free electric field $E(z, t)$ has to satisfy Eq. (65) with $\chi^{(3)} = 0$. Assuming the normalization of $u_l(z)$ in volume $\text{Vol} = Sd$ and the boundary condition $u_l(0) = u_l(d) = 0$, we obtain

$$u_l(z) = \sqrt{\frac{2}{Sd}} \sin \frac{l\pi z}{d} \quad \text{for } l = \pm 1, \pm 2, \dots \quad (71)$$

Substituting Eqs. (69), (70), and (71) into \mathcal{H}_0 , we obtain

$$\mathcal{H}_0 = \sum_{l=\pm 1, \pm 2, \dots} \mathcal{H}_l, \quad (72)$$

$$\mathcal{H}_l = \frac{1}{2}(\dot{q}_l^2 + \omega_l^2 q_l^2), \quad (73)$$

where

$$\omega_l = \sqrt{\frac{1}{\mu_0 \epsilon}} \frac{|l|\pi}{d}. \quad (74)$$

Let us define a new variable $p_l \equiv \dot{q}_l$. Then, we can rewrite \mathcal{H}_l as $\mathcal{H}_l = (1/2)(p_l^2 + \omega_l^2 q_l^2)$, and we obtain the relations $\partial \mathcal{H}_l / \partial q_l = -\dot{p}_l$ and $\partial \mathcal{H}_l / \partial p_l = \dot{q}_l$. Thus, we can regard q_l and p_l as canonical conjugate variables. We perform the quantization of the field by taking the commutation relation $[q_l, p_m] = i\hbar \delta_{lm}$. This commutation relation corresponds with the following relations:

$$q_l = \sqrt{\frac{\hbar}{2\omega_l}} (a_l^\dagger + a_l), \quad (75)$$

$$p_l = i\sqrt{\frac{\hbar\omega_l}{2}} (a_l^\dagger - a_l), \quad (76)$$

$[a_l, a_m^\dagger] = \delta_{lm}$, and $[a_l, a_m] = [a_l^\dagger, a_m^\dagger] = 0$. Rewriting \mathcal{H}_l with the creation and annihilation operators a_l^\dagger and a_l as

$$\mathcal{H}_l = \hbar\omega_l (a_l^\dagger a_l + \frac{1}{2}), \quad (77)$$

we accomplish the quantization of the free field.

Next, we derive the effective Hamiltonian of the quantized field for V as the first-order perturbed energy. We substitute Eqs. (69), (71), and (76) into V , and we obtain

$$\begin{aligned} V &= \frac{1}{2} \chi^{(3)} \frac{1}{\epsilon^2} \left(\frac{2}{Sd} \right)^2 \sum_{k, l, m, n = \pm 1, \pm 2, \dots} p_k p_l p_m p_n \\ &\quad \times \int_{Sd} d^3 \mathbf{x} \sin \frac{k\pi z}{d} \sin \frac{l\pi z}{d} \sin \frac{m\pi z}{d} \sin \frac{n\pi z}{d}. \end{aligned} \quad (78)$$

Using the following formula,

$$\begin{aligned}
& \int_0^d dz \sin \frac{k\pi z}{d} \sin \frac{l\pi z}{d} \sin \frac{m\pi z}{d} \sin \frac{n\pi z}{d} \\
&= \frac{d}{8} (-\delta_{k,l+m+n} + \delta_{k,-l+m+n} + \delta_{k,l-m+n} - \delta_{k,-l-m+n} \\
&\quad + \delta_{k,l+m-n} - \delta_{k,-l+m-n} - \delta_{k,l-m-n} + \delta_{k,-l-m-n}),
\end{aligned} \tag{79}$$

we obtain

$$\begin{aligned}
V &= \frac{\chi^{(3)}}{4\epsilon^2 Sd} \sum_{k,l,m,n=\pm 1, \pm 2, \dots} p_k p_l p_m p_n \\
&\quad \times (-4\delta_{k,l+m+n} + 3\delta_{k+l,m+n} + \delta_{k+l+m+n,0}).
\end{aligned} \tag{80}$$

Here, we assume that we inject only photons of a certain mode l into the material. Thus, terms that contain a_k ($k \neq l$) vanish when we apply them to the state vector. Moreover, we neglect events where photons of mode $k (\neq l)$ are created. Thus, for example, we ignore the case where three photons of ω_l are annihilated and one photon of $\omega_{3l} (= 3\omega_l)$ is created. This implies that the effective terms of V cannot contain a_k^\dagger ($k \neq l$), as well. Hence, the effective potential for photons of mode l is given by

$$\frac{3}{4} \frac{\chi^{(3)}}{\epsilon^2 Sd} \left(\frac{\hbar\omega_l}{2} \right)^2 (a_l^\dagger - a_l)^4. \tag{81}$$

From some calculations, we obtain

$$\begin{aligned}
& (a_l^\dagger - a_l)^4 \\
&= 6(a_l^\dagger)^2 a_l^2 + 8a_l^\dagger a_l + 1 \\
&\quad + (a_l^\dagger)^4 + a_l^4 - 4[(a_l^\dagger)^2 + a_l^2 + (a_l^\dagger)^3 a_l + a_l^\dagger a_l^3].
\end{aligned} \tag{82}$$

Here, we concentrate on the terms which conserve the number of photons. Hence, we can write the effective Hamiltonian as follows:

$$\begin{aligned}
\mathcal{H} &= \hbar\omega_l (a_l^\dagger a_l + \frac{1}{2}) \\
&\quad + \frac{3}{4} \frac{\chi^{(3)}}{\epsilon^2 Sd} \left(\frac{\hbar\omega_l}{2} \right)^2 [6(a_l^\dagger)^2 a_l^2 + 8a_l^\dagger a_l + 1] \\
&= \tilde{\mathcal{H}}_l + \tilde{V}_l,
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
\tilde{\mathcal{H}}_l &= \hbar\omega_l (a_l^\dagger a_l + \frac{1}{2}) \\
&\quad + \frac{3}{4} \frac{\chi^{(3)}}{\epsilon^2 Sd} \left(\frac{\hbar\omega_l}{2} \right)^2 [8a_l^\dagger a_l + 1], \\
\tilde{V}_l &= \frac{9}{8} \frac{\chi^{(3)} \hbar^2 \omega_l^2}{\epsilon^2 Sd} (a_l^\dagger)^2 a_l^2.
\end{aligned} \tag{84}$$

When we estimate the phase rotation induced by Kerr nonlinear interaction, \tilde{V}_l makes a main contribution and $\tilde{\mathcal{H}}_l$ causes only a background phase rotation. Furthermore, in most cases, the relation $\omega_l \gg |\chi^{(3)}|\hbar\omega_l^2/(\epsilon^2 Sd)$ is satisfied and we can take $\tilde{\mathcal{H}}_l \simeq \mathcal{H}_l$. Hence, we obtain the following effective Hamiltonian:

$$\mathcal{H} = \mathcal{H}_l + V_l, \quad (85)$$

where

$$\begin{aligned} \mathcal{H}_l &= \hbar\omega_l(a_l^\dagger a_l + \frac{1}{2}), \\ V_l &= \frac{9}{8} \frac{\chi^{(3)}\hbar^2\omega_l^2}{\epsilon^2 Sd} (a_l^\dagger)^2 a_l^2. \end{aligned} \quad (86)$$

References

- [1] P.W. Shor, ‘Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer’, *SIAM J. Comput.* **26**, 1484–1509 (1997).
- [2] M.A. Nielsen and I.L. Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, U.K., 2000), Chap. 4.
- [3] T. Sleator and H. Weinfurter, ‘Realizable universal quantum logic gates’, *Phys. Rev. Lett.* **74**, 4087–4090 (1995).
- [4] A. Barenco, C.H. Bennett, R. Cleve, D.P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J.A. Smolin, and H. Weinfurter, ‘Elementary gates for quantum computation’, *Phys. Rev. A* **52**, 3457–3467 (1995).
- [5] J.I. Cirac and P. Zoller, ‘Quantum computations with cold trapped ions’, *Phys. Rev. Lett.* **74**, 4091–4094 (1995).
- [6] C. Monroe, D.M. Meekhof, B.E. King, W.M. Itano, and D.J. Wineland, ‘Demonstration of a fundamental quantum logic gate’, *Phys. Rev. Lett.* **75**, 4714–4717 (1995).
- [7] Q.A. Turchette, C.J. Hood, W. Lange, H. Mabuchi, and H.J. Kimble, ‘Measurement of conditional phase shifts for quantum logic’, *Phys. Rev. Lett.* **75**, 4710–4713 (1995).
- [8] N.A. Gershenfeld and I.L. Chuang, ‘Bulk spin-resonance quantum computation’, *Science* **275**, 350–356 (1997).
- [9] E. Knill, R. Laflamme, and G.J. Milburn, ‘A scheme for efficient quantum computation with linear optics’, *Nature* **409**, 46–52 (2001).
- [10] T.C. Ralph, A.G. White, W.J. Munro, and G.J. Milburn, ‘Simple scheme for efficient linear optics quantum gates’, *Phys. Rev. A* **65**, 012314 (2001).
- [11] P. Kok, W.J. Munro, K. Nemoto, T.C. Ralph, J.P. Dowling and G.J. Milburn, ‘Linear optical quantum computing with photonic qubits’, *Rev. Mod. Phys.* **79**, 135–174 (2007).

- [12] E. Yablonovitch, ‘Inhibited spontaneous emission in solid-state physics and electronics’, Phys. Rev. Lett. **58**, 2059–2062 (1987).
- [13] S. John, ‘Strong localization of photons in certain disordered dielectric superlattices’, Phys. Rev. Lett. **58**, 2486–2489 (1987).
- [14] E. Yablonovitch, ‘Photonic band-gap structures’, J. Opt. Soc. Am. B **10**, 283–295 (1993).
- [15] I.L. Chuang and Y. Yamamoto, ‘Simple quantum computer’, Phys. Rev. A **52**, 3489–3496 (1995).
- [16] S.I. Inoue and Y. Aoyagi, ‘Design and fabrication of two-dimensional photonic crystals with predetermined nonlinear optical properties’, Phys. Rev. Lett. **94**, 103904 (2005).
- [17] M. Scalora, J.P. Dowling, C.M. Bowden, and M.J. Bloemer, ‘Optical limiting and switching of ultrashort pulses in nonlinear photonic band gap materials’, Phys. Rev. Lett. **73**, 1368–1371 (1994).
- [18] A. Huttunen and P. Törmä, ‘Band structures for nonlinear photonic crystals’, J. Appl. Phys. **91**, 3988–3991 (2002).
- [19] J. Fiurášek, L. Mišta, Jr., and R. Filip, ‘Entanglement concentration of continuous-variable quantum states’, Phys. Rev. A **67**, 022304 (2003).
- [20] K. Nemoto and W.J. Munro, ‘Nearly deterministic linear optical controlled-NOT gate’, Phys. Rev. Lett. **93**, 250502 (2004).
- [21] W.J. Munro, K. Nemoto, and T.P. Spiller, ‘Weak nonlinearities: a new route to optical quantum computation’, New J. Phys. **7**, 137 (2005).
- [22] J. Lee, M. Paternostro, C. Ogden, Y.W. Cheong, S. Bose, and M.S. Kim, ‘Cross-Kerr-based information transfer processes’, New J. Phys. **8**, 23 (2006).
- [23] H. Jeong, ‘Quantum computation using weak nonlinearities: Robustness against decoherence’, Phys. Rev. A **73**, 052320 (2006).
- [24] P.D. Drummond and D.F. Walls, ‘Quantum theory of optical bistability. I: Nonlinear polarisability model’, J. Phys. A: Math. Gen. **13**, 725–741 (1980).
- [25] L. Mandel and E. Wolf, *Optical coherence and quantum optics* (Cambridge University Press, Cambridge, U.K., 1995), Chap. 22.
- [26] D.A.B. Miller, D.S. Chemla, D.J. Eilenberger, P.W. Smith, A.C. Gossard, and W.T. Tsang, ‘Large room-temperature optical nonlinearity in GaAs/Ga_{1-x}Al_xAs multiple quantum well structures’, Appl. Phys. Lett. **41**, 679–681 (1982).
- [27] D.A.B. Miller, D.S. Chemla, D.J. Eilenberger, P.W. Smith, A.C. Gossard, and W. Wiegmann, ‘Degenerate four-wave mixing in room-temperature GaAs/GaAlAs multiple quantum well structures’, Appl. Phys. Lett. **42**, 925–927 (1983).

- [28] D.S. Chemla, D.A.B. Miller, and P.W. Smith, ‘Nonlinear optical properties of GaAs/GaAlAs multiple quantum well material: phenomena and applications’, *Opt. Eng.* **24**, 556–564 (1985).
- [29] A. Yariv, *Optical electronics in modern communications*, 5th ed. (Oxford University Press, Oxford, U.K., 1997).
- [30] R. de L. Kronig and W.G. Penney, ‘Quantum mechanics of electrons in crystal lattices’, *Proc. Roy. Soc. (London) A* **130**, 499–513 (1931).
- [31] C. Kittel, *Introduction to solid state physics*, 7th ed. (John Wiley & Sons, Inc., New York, 1996), Chap. 7.
- [32] S. Noda, N. Yamamoto, M. Imada, H. Kobayashi, and M. Okano, ‘Alignment and stacking of semiconductor photonic bandgaps by wafer-fusion’, *J. Lightwave Technol.* **17**, 1948–1955 (1999).
- [33] S. Noda, K. Tomoda, N. Yamamoto, and A. Chutinan, ‘Full three-dimensional photonic bandgap crystals at near-infrared wavelengths’, *Science* **289**, 604–606 (2000).